Continuing Results on Norm Based Nonlinear Observers With Examples

Monty J. Smith
Arkansas Tech University, Russellville, Arkansas, 72801

Abstract –Recent new advances in observer theory have been proposed based on nonlinear transformations of the original state. The estimator structure being considered uses state space methods, but allows the observer to approach the original state based on a single scalar parameter. This approach appears to have assets, particularly when the measurement error has an unknown but bounded uncertainty being processed through the measurement array. In these cases, the peaking phenomena due to the feedback gain (as normally witnessed from using the Luenberger observer) appear to be reduced significantly. In this note, emphasis will be given to simulations of a robotics example, but the theoretical results are applicable to numerous aerospace, electrical, and mechanical systems where state estimation is involved. The performance of the estimator considered in this note will be purely based from performance robustness to measurement bias.

I. Introduction

INTERNAL dynamics of a system modeled from aerospace, robotics, structures, electronics, and numerous other physical phenomena can generally be mathematically reconstructed in real time. These model based reconstruction strategies of the internal dynamics are generally derived from input-output mapping of the signals entering and exiting the process or system. In addition, some knowledge of the errors in the model parameters (possibly bounds) and the exogenous and measurement uncertainties (typically statistical based) are assumed known (see [6,7,8,9] for interesting examples). The typical format for the construction of these 'observers' are based upon a state space methodology, thus in general a first order vector based format. There are numerous technical references that convey a good explanation of how this process evolves, but the mathematical approach presented in references [1-3] provides the reader sufficient information to follow the remaining portion of this technical note. The process of developing a state estimator doesn't necessarily imply that the process or system is linear, and more often than not the modeling and estimation process involves a nonlinear system. However, since the work that is being exhibited in this paper is fairly preliminary, the structure of the mathematics will concentrate on linear dynamical systems. The work is not limited to linear systems, but it provides a better view as to the assets in administering such an approach for the construction of a norm based state estimator.

In this note, we will consider the state space based linear system modeled as a first order process,

$$(\Sigma_1) \quad \dot{x} = A\,x + B\,u + Dw\,, \quad z = M\;x + v \tag{1}$$

where $x \in \Re^n$ are coordinates in state space with initial states at time zero designated by $x(0) = x_0$, the known time invariant state space realization $\{A, B, M\}$ with the matrix A assumed Hurwitz, the measurement vector $z \in \Re^z$, and measurement uncertainties denoted by the signal $\{v\}$, the exogenous disturbance {w}, and the notation $\dot{x} = dx/dt$ denoting the time derivative. In addition, the realization is assumed detectable, and since there is no forwarding term u, the system is strictly proper (for convenience of presentation). In this note the disturbance w(t) will always be a zero mean with bounded covariance, and the measurement uncertainty given as any bounded time varying signal, i.e., $\|v\| \le \gamma_v$, where the notation $\|\cdot\|$ denotes the space of piecewise continuous, bounded signals (L_∞ norm), i.e., if a time varying vector $u(t)$ is in L_∞, then $\|u\| = \sup_{t \ge 0} \|u(t)\|_2 \prec \infty$. In addition, the notation $\|C\|_2 = (\lambda_{\max}(C'C))^{1/2}$ is consistent with the literature for any constant matrix $C \in \Re_{mxn}$.

Assistant Professor, Mechanical Engineering Department, (email: msmith89@atu.edu), Senior Member

II. Estimation / Brief Review

Similar to the format given in (1), assume that the following system replicates the system (Σ_1) with the exception that the measurement uncertainty is assumed zero mean without the bias, and there exists an exogenous system disturbance,

$$(\Sigma_2) \quad \dot{x} = A\,x + B\,u + D\,w, \quad z = M\;x + v \tag{2}$$

From consulting the literature on estimation theory (for example, [1-3]) and denoting the estimated state and measurement x_2 and z_2, respectively, the state x can be reconstructed by forming the following realization,

$$\begin{aligned}(\Sigma_3)\;\; \dot{x}_2 &= Ax_2 + Bu + F(z - z_2) \quad z = Mx + v \\ &= Ax_2 + Bu + FMx_2 + Fv - FMx_2,\; z_2 = Mx_2\end{aligned} \tag{3}$$

where F is constructed to assure the matrix $(A - FM)$ is Hurwitz. By defining $\tilde{x}_2 = x_2 - x$ as the error between the state x and the estimated state x_2, the first derivative is given by

$$\begin{aligned}\dot{\tilde{x}}_2 &= (A - FM)\tilde{x}_2 + Fv - Dw \\ &= A\tilde{x}_2 - FMx_2 + Fz - Dw.\end{aligned} \tag{4}$$

This structure provides a foundation for recent developments in estimation theory ([5]), provided the exogenous disturbance w is negligible compared to the measurement error v. This case is not an extreme limitation in many aerospace systems, and will be used as a basis for the continuing work that follows.

III. Estimator in Transformed Coordinates / Review

The main theoretical result that follows enables the construction of the state vector x_p, where x_p satisfies the constraint

$$\lim_{t\to\infty} x_p - x = \|x_2 - x\|^{2/p-1}(x_2 - x) = \|\tilde{x}_2\|^{2/p-1}\tilde{x}_2 \tag{5}$$

and is provided by the following Theorem 1(as presented in [5]).

<u>*Theorem 1*</u>*Consider the linear time invariant stable system with sensor noise, i.e., defined by the following realization*

$$\dot{x} = A\,x + B\,u\,, \quad z = M\;x + v \tag{6}$$

and a pre-constructed linear estimator (designed without consideration of sensor bias),

$$\dot{x}_2 = Ax_2 + Bu + F(z - z_2) \quad z_2 = Mx_2\,, \tag{7}$$

where the matrix $(A - FM)$ is Hurwitz. Then assuming that a predetermined positive scalar q has been chosen such that the bound holds

$$\|q\,\tilde{x}_2(t)\| = \|q\{\int_0^t e^{(A-FM)(t-\tau)}Fv(\tau)d\tau\}\| < 1 \;\forall\; t \ge t_1 \tag{8}$$

for the sensor bias noise source v ,then there exists a nonlinear state estimator x_p satisfying the limit as $t \to \infty$, $x_p - x \to 0$ Furthermore, recalling that $\dot{\tilde{x}}_2 = \dot{x}_2 - \dot{x} = A\tilde{x}_2 - FMx_2 + Fz$, then the structure of this particular state estimator is given by

$$\dot{x}_p = A\,x_p + B\,u - A\|\,q\,\tilde{x}_2\|^{2/p-1}\tilde{x}_2 + \frac{d}{dt}\{\|q\,\tilde{x}_2\|^{2/p-1}q\,\tilde{x}_2\} \tag{9}$$

and converges for any given initial condition pair $\{x_p(0), \tilde{x}_2(0)\}$.

Recall that the positive scalar quantity q has to be chosen to assure $\|q\tilde{x}_2\| < 1$ for all time. In essence,

$$\|q\,\tilde{x}_2(t)\| = \sup_{t\to\infty}\|q\{\int_0^t e^{(A-FM)(t-\tau)}F\,v(\tau)d\tau\}\| < 1 \tag{10}$$

This inequality plays an important role in the development of sufficient conditions that enable performance bounds to be achieved. The next section summarizes these results based on further research and some interesting results restated from the literature.

IV. Performance Bounds and Discussion

There are results that guarantee the bound $\|q\tilde{x}_2\| < 1$ when $\|v\| < \gamma_v$ for an appropriately chosen q. The following Lemma provides the tools for computing q, and technically provides a means for weighting the L_∞ gain from the input signal $v(t)$ to the output error state $\tilde{x}_2(t)$ (for example, see [4,11]).

<u>Lemma 1</u>*For the linear time invariant system*

$$\dot{\tilde{x}} = (A - FM)\tilde{x} + Fv, \quad \tilde{x}(0) = 0 \tag{11}$$

If the matrix (A-FM) is Hurwitz, then the system is finite-gain L_∞ *stable. Moreover,*

$$\|\tilde{x}_2\| \le g_F\,\|v\| \;\; \textit{where} \;\; g_F = \frac{2\lambda^2{}_{\max}(P)\|F\|_2}{\lambda_{\min}(P)} \tag{12}$$

and P is the solution to the Lyapunov equation

$$P(A - FM) + (A - FM)'P + I = 0\cdot \tag{13}$$

<u>Choice of Weight</u> From applying the results of the previous Lemma, the unit bound can be assured on the state estimator error, i.e.,

$$q < 1/(g_F\gamma_v) \;\; \rightarrow \;\; \|q\tilde{x}_2\| \prec 1 \tag{14}$$

hence providing a means for guaranteeing that $\lim x_p - x \to 0$ as $p \to 0$ can be achieved. Although these bounds can be conservative, they provide sufficient conditions for the construction of the nonlinear estimation filter (the state x_p).

<u>Comment on the L-2 gain</u> It's interesting to note that the finite gain on L_2 (the H_∞ problem in the literature, [11]) could also be used as performance indices. However, the transformation and state metric used in this note are instantaneous values in time, and the map used in this presentation is bounded input-bounded output stable. The L_2 gain (to be investigated in future work) assumes the input and output signals are square sumable / finite energy, and hence further research is warranted to determine the assets in using those performance indices. To elaborate, notice that the inequality in (10) can be restated in the frequency domain for expressing the L_2 gain. That is, the constraint for the computation of the weight q becomes

$$\|q\tilde{x}_2\|_2 = q\|G(s)\|_\infty \|v\|_2 \prec 1 \tag{15}$$

where $G(s) = (sI - (A - FM))^{-1}F$ is the stable transfer function equivalent of the estimator realization, and the nomenclature $\|v\|_2$ now denotes the L_2 norm denoted by

$$\|v\|_2 = \sqrt{\int_0^\infty v'(t)v(t)\,dt} \prec \infty \tag{16}$$

Thus, we can always satisfy the bound in (10) by choosing an appropriate q such that

$$q \prec \frac{1}{\|G(s)\|_\infty \|v\|_2} \tag{17}$$

This is an easy computation due to the choice of $\gamma_v = \{\|v\|_2\}$, and the norm $\|G(s)\|_\infty$ is easily computable. Also recall that $\|G(s)\|_\infty < \gamma$ if and only if the same γ, there exists a solution P to the following algebraic Riccati equation

$$P(A - FM) + (A - FM)'P + \gamma^{-2}PFF'P + I = 0 \tag{18}$$

such that $(A - FM + \gamma^{-2}FF'P)$ is asymptotically stable (see [11] for further information). Since the L_2 problem assumes that $\|v\|_2$ is finite, the physical offset of the sensor bias would necessarily have to occur over a finite time interval to be applicable to this solution. In addition, the output norm performance would then become

$$\|x_p - x\|_2 = \|q(x_2 - x)\|_2^{2/p-1}\|q(x_2 - x)\|_2 = \|q\tilde{x}_2\|_2^{2/p} \tag{19}$$

which implies that $\lim x_p - x \to 0$ as $p \to 0$ can still be achieved using the L_2 gain. We conclude with some examples that are based on finite-gain L_∞ algorithms with performance satisfying the inequalities of Lemma 1.

V. Algorithms For Applications

The numerical strategy can be basically restated from [5], with some additional insight provided to exhibit a simpler form for the limiting case when $p \to 0$. Recall that the original set of coupled equations has to be integrated to obtain the newly defined state x_p, as provided by the following algorithm.

Numerical Strategy A nonlinear state x_p can be constructed by simultaneous integration of the following three nonlinear equations,

$$\begin{aligned}
\dot{x}_p &= A x_p + Bu - A\|q\tilde{x}_2\|^{2/p-1} q\tilde{x}_2 + \frac{d}{dt}\{\|q\tilde{x}_2\|^{2/p-1} q\tilde{x}_2\} \\
\dot{\tilde{x}}_2 &= A\tilde{x}_2 - FMx_2 + Fz \\
\dot{x}_2 &= Ax_2 + Bu + F(z - z_2) \quad z_2 = Mx_2,
\end{aligned} \tag{20}$$

where the initial conditions can be independently selected. For completion of presentation, the last derivative as expressed in (20) has the more explicit characterization as follows,

$$\frac{d}{dt}\{\|q\tilde{x}_2\|^{2/p-1} q\tilde{x}_2\} = q^{2/p}\|\tilde{x}_2\|^{2/p-1}\left\{I + (2/p-1)\frac{\tilde{x}_2\,\tilde{x}_2'}{\|\tilde{x}_2\|^2}\right\}\dot{\tilde{x}}_2 \tag{21}$$

with $\dot{\tilde{x}}_2$, as previously given in (20). An interesting assessment of the three dynamic equations in (20) shows that $x_p - x \to 0$ as $p \to 0$ for a sufficiently large time. However, it's interesting to note that for this case, x_p can be constructed directly from the states x_2 and $\tilde{x}_2$ as the following lemma states.

Lemma 2 For the limiting design case where the parameter $p \to 0$, the state x_p can be directly constructed from a direct sum of the states x_2 and $\tilde{x}_2$. That is, $x_p = x_2 - \tilde{x}_2$.

Proof

$\lim_{t\to\infty}(x_p - x) \to \|q(x_2 - x)\|^{2/p-1} q(x_2 - x) = \|q\tilde{x}_2\|^{2/p-1} q\tilde{x}_2$ *and since* $\lim_{t\to\infty} x_2 - x \to \tilde{x}_2$, *we have* $x_p - \|q\tilde{x}_2\|^{2/p-1} q\tilde{x}_2 \to x_2 - \tilde{x}_2$ *which implies that* $\lim_{t\to\infty,\, p\to 0} x_p \to x_2 - \tilde{x}_2$.

The results of Lemma 2 show that the newly constructed state x_p approaches x, although the measurement array z is still being fed back through the estimator nonlinear setup, as seen in equation (20). In fact, the gain F is fixed and hence the bandwidth of the closed loop estimated realization x_p appears reduced from observation of the simulations and plots that follow in section 6.

VI. Flexible Robot Example

A simple example was chosen that represents a single-link manipulator with flexible joints, and taken from [10], with the addition of damping and linearization. The equations representing the system are two coupled second order systems with an externally driven torque and exogenous noise source as follows,

$$\begin{aligned}
&I\ddot{q}_1 + c\dot{q}_1 + MgL\,q_1 + k(q_1 - q_2) = w_1 \\
&J\ddot{q}_2 + c\dot{q}_2 - k(q_1 - q_2) = u + w_2 \\
&z = [q_1\ q_2]' + [v_1\ v_2]'.
\end{aligned} \tag{22}$$

In this example, I and J are moments of inertia, k is a positive spring constant, M is a total mass, L is a distance, c a viscous damping coefficient, and the angular position coordinates are $\{q_1, q_2\}$ with external torque commanded by u, and the exogenous noise pair $\{w_1, w_2\}$, and the measurement noise (bounded signals but continuous) are denoted by $\{v_1, v_2\}$.Although the theoretical results assume zero exogenous noise, the examples included have negligible external disturbances in the simulations. By assigning the states $\vec{x}' = \{x_1, x_2, x_3, x_4\} = \{q_1, \dot{q}_1, q_2, \dot{q}_2\}$, and assuming that the available states for measurements are $z = \{q_1, q_2\}' + \{v_1, v_2\}'$, the first order form becomes

$$\dot{\bar{x}} = A\bar{x} + Bu + D_w\bar{w},\ z = M\bar{x} + \bar{v} \tag{23}$$

where the matrices $\{A, B, D_w, M\}$ are given by

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k+MgL)/I & -c/I & -k/I & 0 \\ 0 & 0 & 0 & 1 \\ k/J & 0 & -k/J & -c/J \end{bmatrix} \quad B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J \end{bmatrix}, \quad D_w=\begin{bmatrix} 0 & 0 \\ 1/I & 0 \\ 0 & 0 \\ 0 & 1/J \end{bmatrix}, \quad M=[1\ \ 0\ \ 1\ \ 0] \tag{24}$$

The numbers assigned for the time invariant realization elements are as follows.

Table 1

Equation Coefficient	Simulation Number Assigned
M	5 (lbm)
L	1 (ft)
I	1(ft lbf s^2)
J	1(ft lbf s^2)
k	64 (ft lbf)
g	32.2 (ft/ s^2)
c	0.1(ft lbf s)

The open loop plant eigenvalues and closed loop estimation eigenvalues are listed in the following Table 2 (the estimation feedback gain F was assigned the value $F=[6.4 \quad 98.6 \quad 9.8 \quad 31.4]$ based on a standard linear quadratic form).

Table 2

Open Loop Eigenvalue	Closed Loop Eigenvalue
-0.05+- 15.73i	-1.65+- 15.71i
-0.05+- 6.45i	-6.51+- 9.35i

Throughout the simulations, the driving external torque command was assigned the function $u(t)=0.2\cdot\sin(2\pi ft)$ where the driving frequency f was fixed at 0.6 Hertz.

For consistency, there were three simulations with three different constants assigned to the parameter p (the transformation constant), $p_{1-3}=\{1.8, 1.2, 0.5\}$. All simulations are shown in following Figure 1-3, with the actual position shown in blue, the data in red is the Luenberger filter results of estimated position, and the black signal represents the estimated position from the nonlinear filter. In all simulations, an estimation bias v of magnitude 0.1 radians was introduced onto the measurement signal ($z(t)=Mx(t)+v$) where the bias v was held at the constant dc offset from $4<t<6$ seconds.

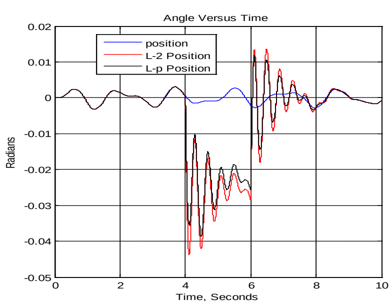

Figure 1 (p = 1.8)

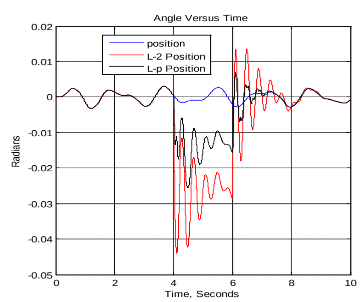

Figure 2 (p = 1.2)

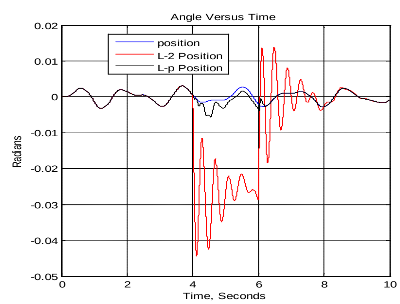

Figure 3 (p = 0.5)

From observation of Figures 1-3, as p approaches zero, the estimation error reduces with respect to the ratio set by the norm coefficient, that is,

$$\lim_{t\to\infty} x_p - x = \|x_2 - x\|^{2/p-1}(x_2 - x) = \|\tilde{x}_2\|^{2/p-1}\tilde{x}_2. \tag{25}$$

Thus, for a given p the ratio of reduction in error is proportional to the linear estimation error as predicted. The state initial conditions were all set to zero in the simulations, although not a necessary requirement due to the Hurwitz criteria.

Discussion on weighting q During the simulation of the plant and estimation process, the value for q (see equation (14)) was computed and consistently gave us an extremely conservative weighting factor to assure that the value for $\|q\,\tilde{x}_2\| < 1$. For example, for all the simulations accomplished that are shown in Figures 1-3, the values for γ_v were constrained in all simulations with an upper bound of 0.1 (i.e., $\|v\| < \gamma_v$). In addition, the spectral content of the Lyapunov solution P had a fairly broad range which resulted in an extremely conservative bound to assure that $q < 1/(g_F \gamma_v) \rightarrow \|q\,\tilde{x}_2\| \prec 1$. Inherently, since the eigenvalues of P were computed to an approximation of values {0.04, 0.14, 13.14, 52.40}, the bound $(g_F \gamma_v)$ exceeds unity and the q turns out to be extremely conservative, hence q was kept at unity in all examples that follow.

VII. Conclusions

The filter presented in this research has interesting performance results. A promising feature of this new filter is the reduction in the error of the constructed estimated state, when the expectation of getting sensor noise bias is highly probable. As seen from the plots, the standard Luenberger observer is sensitive to impulsive errors shortly after the sensor bias is introduced into the system. In this regard, the nonlinear estimator is robust to these disturbances originating from sensor noise, which appears to be predominately due to the design criteria explicit in the nonlinear transformation itself.

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